Rigidity of stable cylinders in three-manifolds

José M. Espinar [*]

[*] Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain;
e-mail: jespinar@ugr.es

Abstract

In this paper we show how the existence of a certain stable cylinder determines (locally) the ambient manifold where it is immersed. This cylinder has to verify a *bifurcation phenomena*, we make this explicit in the introduction. In particular, the existence of such a stable cylinder implies that the ambient manifold has infinite volume.

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1 Introduction

A stable compact domain Σ on a minimal surface in a Riemannian three-manifold $\mathcal{M}$, is one whose area can not be decreased up to second order by a variation of the domain leaving the boundary fixed. Stable oriented domains Σ are characterized by the *stability inequality* for normal variations ψN [11]

$$\int_\Sigma \psi^2 |A|^2 + \int_\Sigma \psi^2 \mathrm{Ric}_{\mathcal{M}}(N,N) \leq \int_\Sigma |\nabla \psi|^2$$

for all compactly supported functions $\psi \in H_0^{1,2}(\Sigma)$. Here $|A|^2$ denotes the the square of the length of the second fundamental form of Σ, $\mathrm{Ric}_{\mathcal{M}}(N,N)$ is the Ricci curvature of $\mathcal{M}$ in the direction of the normal N to Σ and ∇ is the gradient w.r.t. the induced metric.

One writes the stability inequality in the form

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{Area}(\Sigma(t)) = -\int_\Sigma \psi L \psi \geq 0,$$

where L is the linearized operator of the mean curvature

$$L = \Delta + |A|^2 + \mathrm{Ric}_{\mathcal{M}}.$$

In terms of L, stability means that $-L$ is nonnegative, i.e., all its eigenvalues are non-negative. Σ is said to have finite index if $-L$ has only finitely many negative eigenvalues.

From the Gauss Equation, one can write the stability operator as $L = \Delta - K + V$, where Δ and K are the Laplacian and Gauss curvature associated to the metric g respectively, and $V := 1/2|A|^2 + S$, where S denotes the scalar curvature associated to the metric g.

The index form of these kind of operators is

$$I(f) = \int_\Sigma \left\{ \|\nabla f\|^2 - V f^2 + K f^2 \right\}$$

where ∇ and $\|\cdot\|$ are the gradient and norm associated to the metric g. Thus, if Σ is stable, we have

$$\int_\Sigma f L f = -I(f) \leq 0,$$

or equivalently

$$\int_\Sigma f^2 (1/2|A|^2 + S) \leq \int_\Sigma \left\{ \|\nabla f\|^2 + K f^2 \right\}. \tag{1.1}$$

In a seminar paper [5], D. Fischer-Colbrie and R. Schoen proved:

> **Theorem A:** *Let $\mathcal{M}$ be a complete oriented three-manifold of non-negative scalar curvature. Let Σ be an oriented complete stable minimal surface in $\mathcal{M}$. If Σ is non-compact, conformally equivalent to the cylinder and the absolute total curvature of Σ is finite, then Σ is flat and totally geodesic.*

And they state [5, Remark 2]: *We feel that the assumption of finite total curvature should not be essential in proving that the cylinder is flat and totally geodesic.*

Recently, this question was partially answered in [3] under the assumption that the positive part of the Gaussian curvature is integrable, i.e. $K^+ := \max\{0, K\} \in L^1(\Sigma)$, and totally answered by M. Reiris [10], he proved:

> **Theorem B:** *Let $\mathcal{M}$ be a complete oriented three-manifold of non-negative scalar curvature. Let Σ be an oriented complete stable minimal surface in $\mathcal{M}$ diffeomorphic to the cylinder, then Σ is flat and totally geodesic.*

Besides, Bray, Brendle and Neves [1] were able of determining the structure of a three-manifold $\mathcal{M}$ under the assumption of the existence of an area minimizing two-sphere. Specifically, they proved:

Theorem C: *Let $\mathcal{M}$ be a compact three-manifold with $\pi_2(\mathcal{M}) \neq 0$. Denote by $\mathcal{F}$ the set of all smooth maps $f : \mathbb{S}^2 \to \mathcal{M}$ which represent a non-trivial element of $\pi_2(\mathcal{M})$. Set*

$$\mathcal{A}(\mathcal{M}) := \inf \left\{ \text{area}(f(\mathbb{S}^2)) \ : \ f \in \mathcal{F} \right\}.$$

Then,

$$\mathcal{A}(\mathcal{M}) \text{inf}_{\mathcal{M}} R \leq 8\pi,$$

where R denotes the scalar curvature of $\mathcal{M}$. Moreover, if the equality holds, then the universal cover of $\mathcal{M}$ is isometric to the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$ up to scaling.

In this paper, we will go further. We will see how the existence of a stable cylinder verifying a *bifurcation phenomena* determines the ambient manifold $\mathcal{M}$. First, let us make clear what we mean by *bifurcation phenomena*:

Definition 1.1. *We say that a complete minimal surface $\Sigma \subset \mathcal{M}$ bifurcates if there exist $\delta > 0$ and a smooth map $u : \Sigma \times (-\delta, \delta) \to \mathbb{R}$ so that*

- *For each $p \in \Sigma$, we have $u(x, 0) = 0$ and $\frac{\partial}{\partial t}|_{t=0} u(p, t) = 1$. Moreover, $u(p, t) \geq 0$ if $t > 0$ and $u(p, t) \leq 0$ if $t < 0$.*
- *For each $t \in (-\delta, \delta)$, the surface*

$$\Sigma_t := \left\{ \exp_p(u(p, t)N(p)) \ : \ p \in \Sigma \right\},$$

is a complete minimal surface. Here, $\exp$ denotes the exponential map in $\mathcal{M}$.

Now, we can state:

Theorem 1.1. *Let $\mathcal{M}$ be a complete oriented Riemannian three-manifold with nonnegative scalar curvature. Assume there exists $\Sigma \subset \mathcal{M}$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then, Σ is flat, totally geodesic and S vanishes along Σ. Moreover, there exists an open set $\mathcal{U} \subset \mathcal{M}$ so that $\mathcal{U}$ is locally isometric to $C \times (-\delta, \delta)$, where C denotes the standard cylinder $\mathbb{S}^1 \times \mathbb{R}$. Also, if any complete stable cylinder in $\mathcal{M}$ bifurcate for an uniform $\delta > 0$, then $\mathcal{M}$ is locally isometric either to $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ is the flat tori).*

We should point out the condition that Σ bifurcates is necessary. In fact, one can construct the following example: Let $\mathrm{C}(-l, l)$ be the right cylinder of height $2l$ and radius 1 endowed with the flat metric. Close it up with two spherical caps S_i, $i = 1, 2$ (one on the top and another on the bottom). Now, smooth the surface $\mathcal{M}^2 = \mathrm{C}(-l, l) \cup S_1 \cup S_2$ so that it is flat on $\mathrm{C}(-l + \varepsilon, l - \varepsilon)$, for some $\varepsilon > 0$, and has nonnegative Gaussian curvature.

Consider the three-manifold $\mathcal{M}^3 = \mathcal{M}^2 \times \mathbb{R}$. One can see that, if we take a closed geodesic $\gamma(t) \subset \mathrm{C}(-l+\varepsilon, l-\varepsilon)$, $t \in (-l+\varepsilon, l-\varepsilon)$, the surface $\Sigma(t) := \gamma(t) \times \mathbb{R}$ is a complete stable minimal cylinder in $\mathcal{M}$ that bifurcates, but, when we reach $t = l - \varepsilon$, this property it might disappear (it could bifurcate as constant mean curvature surfaces at one side, but not minimal).

One interesting consequence of Theorem 1.1 is the following:

Corollary 1.1. *Let $\mathcal{M}$ be a complete oriented Riemannian three-manifold with nonnegative scalar curvature. Assume there exists $\Sigma \subset \mathcal{M}$ a complete stable minimal surface conformally equivalent to a cylinder that bifurcates. Then,*

$$\mathrm{Vol}(\mathcal{M}) = +\infty.$$

Actually, the above conclusion (that is, the above Corollary 1.1) is also valid when the cylinder bifurcates only at one side.

2 Preliminaries

We denote by $\mathcal{M}$ a complete connected orientable Riemannian three-manifold, with Riemannian metric g. Moreover, throughout this work, we will assume that its scalar curvature is nonnegative, i.e., $S \geq 0$. $\Sigma \subset \mathcal{M}$ will be assumed to be connected and oriented.

We denote by N the unit normal vector field along Σ. Let $p_0 \in \Sigma$ be a point of the surface and $D(p_0, s)$, for $s > 0$, denote the geodesic disk centered at p_0 of radius s. We assume that $\overline{D(p_0, s)} \cap \partial\Sigma = \emptyset$. Moreover, let r be the radial distance of a point p in $D(p_0, s)$ to p_0. We write $D(s) = D(p_0, s)$.

We also denote

$$\begin{aligned} l(s) &= \mathrm{Length}(\partial D(s)) \\ a(s) &= \mathrm{Area}(D(s)) \\ K(s) &= \int_{D(s)} K \\ \chi(s) &= \text{Euler characteristic of } D(s). \end{aligned}$$

Let $\Sigma \subset \mathcal{M}$ be a stable minimal surface diffeomorphic to the cylinder, then, from Theorem B [10], Σ is flat and totally geodesic. We will give a (more general) proof of this result in the abstract setting of Schrödinger-type operators:

Lemma 2.1. *Let Σ be a complete Riemannian surface. Let $L = \Delta + V - aK$ be a differential operator on Σ acting on compactly supported $f \in H_0^{1,2}(\Sigma)$, where $a > 1/4$ is constant, $V \geq 0$, Δ and K are the Laplacian and Gauss curvature associated to the metric g respectively.*

Assume that Σ is homeomorphic to the cylinder and $-L$ is non-negative. Then, $V \equiv 0$ and $K \equiv 0$, therefore,

$$\mathrm{Ker}L := \{1\},$$

i.e., its kernel is the constant functions. Here, L denotes the Jacobi operator.

Proof. Set $b \geq 1$ and let us consider the radial function

$$f(r) := \begin{cases} (1 - r/s)^b & r \leq s \\ 0 & r > s \end{cases},$$

where r denotes the radial distance from a point $p_0 \in \Sigma$. Then, from [3, Lemma 3.1] (see also [9]), we have

$$\int_{D(s)} (1 - r/s)^{2b} V \leq 2a\pi G(s) + \frac{b(b(1-4a)+2a)}{s^2} \int_0^s (1 - r/s)^{2b-2} l(r),$$

where

$$G(s) := -\int_0^s (f(r)^2)' \chi(r).$$

Therefore, since $a > 1/4$, we can find $b \geq 1$ so that $b(1-4a)+2a \leq 0$. So

$$\int_{D(s)} (1 - r/s)^{2b} V \leq 2a\pi G(s).$$

- **Step 1:** V vanishes identically on Σ.

  Suppose there exists a point $p_0 \in \Sigma$ so that $V(p_0) > 0$. From now on, we fix the point p_0. Then, there exists $\epsilon > 0$ so that $V(q) \geq \delta$ for all $q \in D(\epsilon) = D(p_0, \epsilon)$. Since Σ is topollogically a cylinder, there exists $s_0 > 0$ so that for all $s > s_0$ we have $\chi(s) \leq 0$ (see [2, Lemma 1.4]).

  Now, from the above considerations, there exists $\beta > 0$ so that

  $$0 < \beta \leq 2a\pi G(s).$$

  But, following [3], we can see that

  $$\begin{aligned} G(s) &= -\int_0^s (f(r)^2)'\chi(r) = -\int_0^{s_0} (f(r)^2)'\chi(r) - \int_{s_0}^s (f(r)^2)'\chi(r) \\ &\leq -\int_0^{s_0} (f(r)^2)' = -\left(f(s_0)^2 - f(0)^2\right) = -f(s_0)^2 + 1 \\ &= -\left(1 - s_0/s\right)^{2b} + 1, \end{aligned}$$

  since $-\int_{s_0}^s (f(r)^2)'\chi(r) \geq 0$. Therefore,

  $$G(s) \leq 1 - (1 - s_0/s)^{2b} \to 0, \text{ as } s \to +\infty,$$

  which is a contradiction. Thus, V vanishes identically along Σ.

- **Step 2:** K vanishes identically on Σ. In particular, Σ is parabolic.

  First, note that $L := \Delta - aK$. From [4], there is a smooth positive function u on Σ such that $Lu = 0$. Set $\alpha := 1/a$. Then, from [9] (following ideas of [4]), the conformal metric $\tilde{ds}^2 := u^{2\alpha} ds^2$, where ds^2 is the metric on Σ, is complete and its Gaussian curvature $\tilde{K}$ of is non-negative, i.e. $\tilde{K} \geq 0$.

  On the one hand, the respective Gaussian curvatures are related by

  $$\alpha \Delta \ln u = K - \tilde{K} u^{2\alpha}.$$

  On the other hand, since Σ is topologically a cylinder, the Cohn-Vossen inequality says

  $$\int_\Sigma \tilde{K} \leq 0,$$

  that is, $\tilde{K}$ vanishes identically.

  Thus, $K = \alpha \Delta \ln u$. From this last equation, we get:

  $$aK = \frac{1}{u} \Delta u - \frac{|\nabla u|^2}{u^2},$$

  that is,

  $$\frac{|\nabla u|^2}{u} = \Delta u - aKu = 0.$$

  This last equation implies that u is constant, and since u satisfies $Lu = 0$, we have that K vanishes identically on Σ. In particular, Σ is parabolic (see [6, Lemma 5])

This implies that the Jacobi operator becomes $L := \Delta$, and so the constant functions are in the kernel. But, since Σ is parabolic, such a kernel has dimension one (see [8]), therefore

$$\mathrm{Ker} L := \{1\}.$$

$\square$

Set C $:= \mathbb{S}^1 \times \mathbb{R}$ the flat cylinder, then we can parametrize Σ as the isometric immersion $\psi_0 : \mathrm{C} \to \mathcal{M}$ where $\Sigma := \psi_0(\mathrm{C})$. Also, set $N_0 : \mathrm{C} \to N\Sigma$ the unit normal vector field along Σ.

Assume Σ bifurcates (see Definition 1.1), then there exist $\delta > 0$ and a smooth map $u : \mathrm{C} \times (-\delta, \delta) \to \mathbb{R}$ so that the surface $\Sigma_t := \psi_t(\mathrm{C})$, $\psi_t : \mathrm{C} \to \mathcal{M}$ where

$$\psi_t(p) := \exp_{\psi_0(p)}(u(p,t) N_0(p)), \; p \in \mathrm{C},$$

is a complete minimal surface.

For each $t \in (-\delta, \delta)$, the lapse function $\rho_t : \Sigma \to \mathbb{R}$ is defined by

$$\rho_t(p) = g\left(N_t(p), \frac{\partial}{\partial t}\psi_t(p)\right).$$

Clearly, $\rho_0(p) = 1$ for all $p \in$ C. Also, the lapse function satisfies the Jacobi equation

$$\Delta_t \rho_t + (\mathrm{Ric}(N_t) + |A_t|^2)\rho_t = 0, \tag{2.1}$$

since ψ_t(C) is minimal for all $|t| < \delta$.

Lemma 2.2. *There exists $0 < \delta' < \delta$ such that Σ_t is a stable minimal surface for each $t \in (-\delta, \delta)$. Thus, Σ_t is flat, totally geodesic and S vanishes along Σ_t for each $t \in (-\delta, \delta)$.*

Proof. First, note that, the lapse function is not negative for all $|t| < \delta$ and therefore, by (2.1) and the Maximum Principle, either ρ_t vanishes identically or $\rho_t > 0$ for each $|t| < \delta$.

So, since

$$\rho_t \to \rho_0 \equiv 1, \text{ as } t \to 0,$$

thus, we can find a uniform constant $0 < \delta' < \delta$ such that $\rho_t > 0$ for all $|t| \leq \delta'$.

Therefore, ρ_t, $|t| \leq \delta'$, is a positive function solving the Jacobi equation. This implies that Σ_t is stable for all $|t| \leq \delta'$ (see [4]).

The last assertion follows from Lemma 2.1 and Σ_t be stable.

$\square$

3 Proof of Theorem 1.1

From Definition 1.1 and Lemma 2.2, there exists $\delta > 0$ so that Σ_t is a complete minimal stable surface, which is flat, totally geodesic and $S = 0$ along Σ_t, for each $|t| < \delta$.

Now, we follows ideas of [1]. Since $\mathrm{Ric}(N_t) + |A_t|^2 \equiv 0$ and $H(t) = 0$ for each $|t| < \delta$, from (2.1) and Σ_t being parabolic, we obtain that ρ_t is constant. Thus, since Σ_t is totally geodesic,

$$\begin{array}{rccc} Y: & \mathrm{C} \times (-\delta, \delta) & \to & \mathcal{M} \\ & (p, t) & \to & Y(p, t) := N_t(p) \end{array}$$

is parallel. Also, the flow of N_t is a unit speed geodesic flow (see [7]). Moreover, the map

$$\begin{array}{rccc} \Phi: & \Sigma \times (-\delta, \delta) & \to & \mathcal{M} \\ & (p, t) & \to & \Phi(p, t) := \exp_{\psi_0(p)}(t\, N(p)) \end{array}$$

is a local isometry onto $\mathcal{U} = \bigcup_{|t|<\delta} \Sigma_t$. Therefore, Φ is a diffeomorphism onto $\mathcal{U}$, which implies that $Y : \mathrm{C} \times (-\delta, \delta) \to \mathcal{U}$ is a globally defined unit Killing vector field. This implies that $\mathcal{U}$ is locally isometric to C $\times (-\delta, \delta)$.

Now, assume that any stable minimal complete cylinder bifurcates for an uniform $\delta > 0$. Then, we can start with a complete stable minimal cylinder Σ_0 that bifurcates, and then by the above considerations, Σ_t, for each $|t| < \delta$, is complete, flat, totally geodesic and S vanishes along Σ_t. Moreover, Σ_t is strongly stable for each $|t| < \delta$. Note that Σ_δ is a strongly stable minimal surfaces conformally equivalent to a cylinder, since it is limit of strongly stable minimal surfaces Σ_t which are flat and totally geodesic, then Σ_δ is totally geodesic, flat and $S = 0$ along Σ_δ. Therefore, by Definition 1.1 and Lemma 2.2, there exists $\delta > 0$ so that Σ_t, $-\delta < t < 2\delta$, is flat, totally geodesic and S vanishes along Σ_t. Continuing this argument, Σ_t is flat, totally geodesic and S vanishes along Σ_t for each $t \in \mathbb{I}$, where $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = \mathbb{S}^1$.

As we did above, since $\mathrm{Ric}(N_t) + |A_t|^2 \equiv 0$ and $H(t) = 0$ for each $t \in \mathbb{I}$, from (2.1) and Σ_t being parabolic, we obtain that ρ_t is constant. Thus, since Σ_t is totally geodesic,

$$\begin{array}{rccc} Y: & \mathrm{C} \times \mathbb{I} & \to & \mathcal{M} \\ & (p,t) & \to & Y(p,t) := N_t(p) \end{array}$$

is parallel, where $\mathbb{I} = \mathbb{R}$ or $\mathbb{I} = \mathbb{S}^1$. Also, the flow of N_t is a unit speed geodesic flow (see [7]). Moreover, the map

$$\begin{array}{rccc} \Phi: & \Sigma \times \mathbb{I} & \to & \mathcal{M} \\ & (p,t) & \to & \Phi(p,t) := \exp_{\psi_0(p)}(t\, N(p)) \end{array}$$

is a local isometry, which implies that it is a covering map. Therefore, Φ is a diffeomorphism, which implies that $Y : \mathrm{C} \times \mathbb{I} \to \mathcal{M}$ is a globally defined unit Killing vector field. This implies that $\mathcal{M}$ is locally isometric either to $\mathbb{S}^1 \times \mathbb{R}^2$ or $\mathbb{T}^2 \times \mathbb{R}$ (here $\mathbb{T}^2$ denotes the flat tori).

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